**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MMAT5000 Analysis I (Fall 2015)**
**Quiz 2**

Time allowed: 90 minutes
Total points: 25 points

1. (a) By using the $\epsilon$-definition, show that $\lim_{n\to\infty} \frac{2n}{n+3} = 2$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

   $$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

   By using the $\delta - \epsilon$ definition, show that $f(x)$ is continuous at $x = 0$.

(8 Points)

2. (a) State without proof the Bolzano-Weierstrass theorem.

   (b) Let $S = \{\sin n : n \in \mathbb{N}\}$. Show that there exists at least one cluster point of $S$.

(5 Points)

3. Let $\{x_n\}$ be a sequence of real numbers.

   (a) Suppose that $\lim_{n\to\infty} x_n = L$, prove that $\lim_{n\to\infty} |x_n| = |L|$.

   (b) Does the converse of the statement hold? Prove your assertion.

(5 Points)

4. A set of real numbers $K$ is said to be compact provided that every sequence in $K$ has a subsequence that converges to a point in $K$.

   Suppose that $K$ is a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ is a continuous function.

   (a) Show that $f$ is bounded above.

   (b) Show that there exists $x_M \in K$ such that $f(x) \leq f(x_M)$ for all $x \in K$.

(7 Points)